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Linear algebra I

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Problem Set # 5

Exercise 1($\star$):
For $A \in M_{n,m}(K)$, we have defined linear operators $L_A : K^n \to K^m$ via $L_A(x) = A.x$. Prove that

1. L_A is linear.
2. $L_{AB} = L_A \circ L_B$, for any $A, B \in M_n(K)$.
3. If $n = m$, L_A is an invertible linear operator if and only if A^{-1} exists in $M_n(K)$ and then $(L_A)^{-1} = (L_{A^{-1}})$
4. With the standard bases in K^n, K^m then $[L_A]_{\mathcal{Y}\mathcal{X}} = A$.

Exercise 2($\star$):
Let $\mathcal{P} = \mathbb{R}[X]$ be the infinite dimensional space of polynomials over $\mathbb{R}$. Consider the linear operators $D : \mathcal{P} \to \mathcal{P}$.

1. Derivative
$$D(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = a_1 + 2a_2x + ... + na_nx^{n-1}$$
2. Antiderivative
$$A(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = a_0x + \frac{a_1}{2}x^2 + ... + \frac{a_n}{n+1}x^{n+1}$$

Show that $D \circ A = Id_{\mathcal{P}}$ but that $A \circ D \neq Id_{\mathcal{P}}$. Show that D is surjective and A is one to one but $ker(D) \neq 0$ and $range(A) \neq \mathcal{P}$.
(Note that this behavior is possible only in infinite dimensional space since otherwise one-to-one, surjective and bijective are equivalent.)
Consider now $\mathcal{P}_n$ the set of the polynomial of degree at most n and let $\mathcal{X} = \{1, x, x^2, ..., x^n\}$ and the restriction $D_n : \mathcal{P}_n \to \mathcal{P}_n$ of D to $\mathcal{P}_n$. Determine

1. Dimensions of $K(D_n) = ker(D_n)$ and $R(D_n) = range(D_n)$.
2. The matrix $[D_n]_{\mathcal{X},\mathcal{X}}$.
3. Find bases for $K(D_n)$ and $R(D_n)$.

Exercise 3($\star$): 4 points
Determine the dimensions and bases for $R(T) = range(T)$ and $K(T) = ker(T)$ for the linear operator $T : \mathbb{C}^4 \to \mathbb{C}^3$ determined by the 3×4 matrix

$$\begin{pmatrix} 2 & 4 & 8 & -2 \\ 0 & 1 & 2 & 4 \\ 1 & 3 & 6 & 3 \end{pmatrix}$$

Exercise 4($\star$):
Let $T : V \to W$ be a linear operator between finite dimensional vector spaces. Prove that there exist bases $\mathcal{X} = \{e_1, ..., e_n\} \subset V$ and $\mathcal{N} = \{f_1, ..., f_m\} \subset W$.

$$[T]_{\mathcal{N},\mathcal{X}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m\times n} \text{ where } I_{r\times r} = \begin{pmatrix} 1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & 1 \end{pmatrix}_{r\times r}$$

where $r = dim(range(T))$, $n = dim(V)$, $m = dim(W)$.
Hint: See the class note.

Exercise 5($\star$):
If E, F are finite dimensional subspaces in a vector space V.

1. Prove that $E + F = \{a + b : a \in E, b \in F\}$ is finite dimensional subspace.
2. Assume that $E \cap F = \{0\}$, we can explicitly compute its dimension. Prove that $dim(E + F) = dim(E) + dim(F)$ for direct sums of subspaces.
   Note, in particular we have $dim(E \oplus F) = dim(E) + dim(F)$ for direct sums of subspaces.
3. Bonus: If E, F are finite dimensional subspaces in a vector space V prove the following general Dimension Formula

$$dim(E + F) = dim(E) + dim(F) - dim(E \cap F)$$